

Constructing reparametrization invariant metrics on spaces of plane curves

Martin Bauer, Martins Bruveris, Colin Cotter, Stephen Marsland, Peter Michor

Abstract—Metrics on shape space are used to describe deformations that take one shape to another, and to determine a distance between them. We study a family of metrics on the space of curves, that includes several recently proposed metrics, for which the metrics are characterised by mappings into vector spaces where geodesics can be easily computed. This family consists of Sobolev-type Riemannian metrics of order one on the space $\text{Imm}(S^1, \mathbb{R}^2)$ of parametrized plane curves and the quotient space $\text{Imm}(S^1, \mathbb{R}^2)/\text{Diff}(S^1)$ of unparametrized curves. For the space of open parametrized curves we find an explicit formula for the geodesic distance and show that the sectional curvatures vanish on the space of parametrized and are non-negative on the space of unparametrized open curves. For the metric, which is induced by the “R-transform”, we provide a numerical algorithm that computes geodesics between unparameterised, closed curves, making use of a constrained formulation that is implemented numerically using the RATTLE algorithm. We illustrate the algorithm with some numerical tests that demonstrate its efficiency and robustness.

Index Terms—curve matching, elastic metric, geodesic shooting, reparametrization group, Riemannian shape analysis, shape space



1 INTRODUCTION

The mathematical analysis of shape has become the focus of intense research interest in recent years, not least because of applications in image analysis and computer vision, where methods based on geodesic active contours or ‘snakes’ are used for segmentation, tracking and object recognition [24, 25]. Another source of applications is biomedical image analysis, where the study and comparison of shapes form a large part of the field of computational anatomy [8, 9].

For the purposes of this paper we define shape to be the image of a smooth closed plane curve. A slightly narrower definition would be to define a shape as the outline of a smooth, simply connected domain in the plane. The latter definition would exclude objects like the figure eight from the class of shapes, which we allow in our definition.

A key problem in shape analysis is to define a distance function between shapes, whose numerical computation is feasible and which can be used to measure similarity between shapes or as the basis for classification of shapes. One way to arrive at a distance function is by equipping the space of shapes with a Riemannian metric. The Riemannian metric allows us to measure the length of paths and we can define the distance between two shapes to be the length of the shortest path connecting them. These shortest paths are called geodesics and the resulting distance function the geodesic distance induced by the Riemannian metric.

A shape can be represented via a smooth closed curve $c : S^1 \rightarrow \mathbb{R}^2$ and two curves c, d represent the same shape, if they have the same image, $c(S^1) = d(S^1)$, or equivalently, if there exists a reparametrization map $\varphi \in \text{Diff}(S^1)$, such that one curve is a reparametrization of the other one, i.e. $c = d \circ \varphi$. Here $\text{Diff}(S^1)$ denotes the

group of smooth invertible maps $\varphi : S^1 \rightarrow S^1$ from the circle onto itself.

We will work in this paper with the class of regular or immersed curves c , that is those, whose tangent vector doesn’t vanish, i.e., $c'(\theta) \neq 0$. Upon the space of immersed curves,

$$\text{Imm}(S^1, \mathbb{R}^2) := \{c \in C^\infty(S^1, \mathbb{R}^2) \mid c'(\theta) \neq 0\},$$

acts the diffeomorphism group $\text{Diff}(S^1)$ from the right via $(\varphi, c) \mapsto c \circ \varphi$. Using this setting we can identify a shape with an equivalence class $[c]$, that is an element of the quotient space $\text{Imm}(S^1, \mathbb{R}^2)/\text{Diff}(S^1)$. This quotient

$$\mathcal{S} = \text{Imm}(S^1, \mathbb{R}^2)/\text{Diff}(S^1)$$

is the shape space of immersed curves modulo reparametrizations.

To arrive at a distance function on shape space two steps are necessary. First we have to define a Riemannian metric on the space of immersed curves. A Riemannian metric is an inner product measuring the length of infinitesimal deformations of a curve. Such deformations h, k are represented by vector fields along the curve c and the inner product, which depends on the curve, is denoted by $G_c(h, k)$. The first step consists of defining the metric and computing its geodesics on the space of immersed curves.

If the metric is invariant under the action of the reparametrization group $\text{Diff}(S^1)$, then it induces a Riemannian metric on shape space \mathcal{S} , which in turn gives rise to the geodesic distance function. The second step is to find the right representatives c, d of the equivalence classes $[c]$ and $[d]$, such that the geodesic distance $\text{dist}^{\text{Imm}}(c, d)$ coincides with $\text{dist}^{\mathcal{S}}([c], [d])$.

1.1 Shape metrics and related work

The simplest reparametrization invariant metric on $\text{Imm}(S^1, \mathbb{R}^2)$ is the L^2 -metric

$$G_c(h, k) = \int_{S^1} \langle h, k \rangle ds,$$

where we integrate over arc-length $ds = |c'(\theta)|d\theta$. However it was found in [16] that the geodesic distance induced by this metric vanishes, i.e. the distance between any two shapes is 0, which renders this metric unsuitable for shape analysis.

A possible way to overcome this is to add terms involving higher derivatives of h and k to the metric, like

$$G_c(h, k) = \int_{S^1} \langle h, k \rangle + A \langle D_s h, D_s k \rangle ds,$$

where $D_s h = \frac{1}{|c'|} h'$ denotes the arc-length derivative of h and $A > 0$ is a constant. This leads to the class of Sobolev-type metrics introduced in [19] and studied further in [3, 21].

Another family of metrics, the almost local metrics [4, 17], prevent the geodesic distance from vanishing by introducing a weight function in the integral. Examples of weight functions involving the curvature or length are $w(\theta) = 1 + A\kappa(\theta)^2$ and $w(\theta) = \ell(c)_1$.

Sobolev-type metrics of arbitrary order were studied in [5, 15, 19]. Although they are a natural generalization of the L^2 -metric from a theoretical point of view, their numerical treatment is rather involved. This stems mainly from the fact that the geodesic equation of a Sobolev-type metric of order k is a highly nonlinear PDE of order $2k$. While there have been some attempts to solve the geodesic equation directly for order 1 for curves [20] and surfaces [2], the metrics of higher order are still practically untouched.

To bypass these difficulties, one can restrict the attention to Sobolev-type metrics of order one, in particular to the family of metrics studied in [20], which are of the form

$$G_c^{a,b}(h, h) = \int_{S^1} a^2 \langle D_s h, n \rangle^2 + b^2 \langle D_s h, v \rangle^2 ds,$$

with constants $a, b \in \mathbb{R}^+$ and with v and n denoting the unit tangent and normal vectors to c . Metrics of these form are called elastic metrics. The term involving the normal vector can be seen as measuring the bending of the curve c under the deformation h , while the other term measures the stretching of c .

Following ideas of [26], it was shown in [27] that it is possible to find explicit formulas for geodesics of the elastic metric, when $b^2 = a^2$. To achieve this a curve c was represented by the square-root of its velocity vector, $\sqrt{c'}$, with c' being interpreted as a complex number. In this representation the $G^{1,1}$ -metric has a particularly simple form which allowed to find explicit formulas for geodesics between two curves as well as the geodesic distance.

A similar motivation underlies the introduction of the R -map in [23]. The R -map is a transformation, which maps a curve c to $R(c) = c'/\sqrt{|c'|}$ with the effect that the $G^{a,b}$ -metric simplifies for $4b^2 = a^2$. While not allowing to write down explicit formulas for geodesics, this map greatly simplifies their numerical computation.

Related to the R -map is the Q -map, introduced in [11], which maps a curve c to $Q(c) = \sqrt{|c'|}c$. The advantage of this map is that it generalizes easily to surfaces. It does however present theoretical difficulties as explained in Section 5.

There are other approaches to define a Riemannian metric on shape space: the Large Deformation Diffeomorphic Metric Mapping approach [8, 6], where a Riemannian metric is induced from the diffeomorphism group $\text{Diff}(\mathbb{R}^2)$ of the ambient space or use of conformal welding [22] to represent shapes as diffeomorphisms of the circle.

1.2 The reparametrization group $\text{Diff}(S^1)$

After equipping $\text{Imm}(S^1, \mathbb{R}^2)$ with a Riemannian metric, there are various possibilities how to perform the minimization over the reparametrization group. The problem is challenging because of the nonlinear nature of both $\text{Diff}(S^1)$ and since all the spaces involved are infinite dimensional.

One approach is to replace the infinite dimensional space $\text{Diff}(S^1)$ by a finite-dimensional one and to perform the minimization over this smaller space. A possible choice for the smaller space, used in [23], are diffeomorphisms φ , for whom the Fourier series of $\sqrt{\varphi'}$ is truncated at a fixed length.

In [6] elements of $\text{Diff}(S^1)$ were generated as flows of vector fields. The advantage of using vector fields is that they form a linear space, which enables us to use gradient-based optimization algorithms. However, due to the regularization terms needed to ensure convergence of the algorithm, the computed distance failed to be symmetric.

In the paper [24] the authors describe an iterative procedure directly on $\text{Diff}(S^1)$ based on geometric considerations. We will expand on this minimization scheme in Section 6.

1.3 Overview of the paper

In this paper we introduce a R -transform, which enables us to represent all elastic metrics satisfying $4b^2 \geq a^2$ as pullbacks of the flat L^2 -metric, generalizing the result of [23], which only applied to $4b^2 = a^2$.

Then we proceed to analyse the mathematical properties of this metric, first on open curves in Section 2, followed by closed curves in Section 3. In particular we derive a formula for the geodesic distance between open curves and show that the space of open curves is flat in the sense of Riemannian geometry. We show in Section 4 how the latter result implies that the sectional curvature on shape space of open curves is non-negative. In

addition we derive a formula for the sectional curvature on the space of closed parametrized curves.

Section 5 is devoted to general shape transformations, that can be used to construct reparametrization invariant metrics on $\text{Imm}(S^1, \mathbb{R}^2)$. Both the Q -map from [11] and the representation of curves used in [27] fit into this framework.

We move on in Section 6 from theoretical considerations to numerical computations. First we propose, using the R -transform developed in Section 2, an efficient shooting algorithm for parametrized curves. Then we show how the method of optimizing over reparametrizations, that was also considered in [24], can be seen as a gradient descent with respect to a certain Riemannian metric on $\text{Diff}(S^1)$.

Finally in Section 7, we show some experiments and identify the limits of the metric for practical use.

2 THE R -TRANSFORM FOR OPEN CURVES

We want to study a family of Riemannian metrics on the space of open and closed curves, which are constructed as pullback metrics of a simpler metric under a transformation of the curve. Let us define the R -transform of a plane curve $c \in \text{Imm}([0, 2\pi], \mathbb{R}^2)$ by

$$R^{a,b} : \text{Imm}([0, 2\pi], \mathbb{R}^2) \rightarrow C^\infty([0, 2\pi], \mathbb{R}^3)$$

$$R^{a,b}(c) = |c'|^{1/2} \left(a \begin{pmatrix} v \\ 0 \end{pmatrix} + \sqrt{4b^2 - a^2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right).$$

Here $a, b \in \mathbb{R}^+$ with $4b^2 \geq a^2$ are positive numbers, whose meaning will become clear, when we compute the pullback metric. We will omit the parameters a, b when the meaning is unambiguous. We denote by $v = \frac{c'}{|c'|}$ the unit length tangent vector of c . The R -transform maps an open plane curve to a space curve. We will see that equipping the space $C^\infty([0, 2\pi], \mathbb{R}^3)$ with a flat L^2 -metric, i.e. considering the vector space $C^\infty([0, 2\pi], \mathbb{R}^3)$ with the L^2 -inner product as a Riemannian manifold, will generate Sobolev metrics of order one via the pullback by the R -transform. For the choice of parameters $4b^2 = a^2$ the R -transform reduces to the R -map as studied in [23]. The metric induced by the $R^{a,b}$ -transform also coincides with the metric studied in [20] corresponding to the parameters \sqrt{a}, \sqrt{b} .

First let us note the following properties of the R -transform.

Lemma 2.1. *For a curve $c \in \text{Imm}([0, 2\pi], \mathbb{R}^2)$ and a diffeomorphism $\varphi \in \text{Diff}^+([0, 2\pi])$ we have:*

- *Equivariance under reparametrizations*

$$R(c \circ \varphi) = |\varphi'|^{1/2} \cdot (R(c) \circ \varphi)$$

- *Translation invariance*

$$R(c + p) = R(c) \text{ for } p \in \mathbb{R}^2$$

- *Scaling property*

$$R(\rho \cdot c) = \rho^{1/2} \cdot R(c) \text{ for } \rho \in \mathbb{R}_{>0}$$

- *Preservation of length under reparametrizations*

$$\int_0^{2\pi} |R(c \circ \varphi)|^2 d\theta = \int_{S^1} |R(c)|^2 d\theta.$$

Proof: We will show the first property, since it will be important for the reparametrization invariance of the pullback metric. From $(c \circ \varphi)' = (c' \circ \varphi)\varphi'$ it follows that the unit length tangent vector v behaves under reparametrizations as

$$v(c \circ \varphi) = \frac{(c \circ \varphi)'}{|(c \circ \varphi)'|} = \frac{c'}{|c'|} \circ \varphi = v(c) \circ \varphi.$$

The term $|c'|^{1/2}$ reparametrizes as $|(c \circ \varphi)'|^{1/2} = |\varphi'|^{1/2} |c'| \circ \varphi^{1/2}$, and we have established the first property. The other properties can be verified by a simple calculation. \square

In the next theorem we calculate the pullback of the flat L^2 -metric under the R -transform.

Theorem 2.2 (The pullback metric on Imm). *The pullback of the L^2 -inner product on $C^\infty([0, 2\pi], \mathbb{R}^3)$ to the manifold of immersions by the $R^{a,b}$ -transform is given by:*

$$G_c^{a,b}(h, h) = \int_0^{2\pi} a^2 \langle D_s h, n \rangle^2 + b^2 \langle D_s h, v \rangle^2 ds,$$

where n is the unit length normal vector to the curve c , the arc-length derivative of h along the curve c is denoted by $D_s h = \frac{1}{|c'|} h'$ and $ds = |c'| d\theta$ is the integration with respect to arc-length. The metric can be written in the form

$$G_c^{a,b}(h, h) = \int_0^{2\pi} \langle P_c^{a,b}(h), h \rangle ds$$

where the associated differential operator $P_c^{a,b}$ is given by:

$$P_c^{a,b}(h) = -a^2 \langle D_s^2 h, n \rangle n - b^2 \langle D_s^2 h, v \rangle v$$

$$+ (a^2 - b^2) \kappa (\langle D_s h, v \rangle n + \langle D_s h, n \rangle v)$$

$$+ (\delta_{2\pi} - \delta_0) (a^2 \langle n, D_s h \rangle n + b^2 \langle v, D_s h \rangle v),$$

with κ being the curvature of the curve.

The pullback defines a Riemannian metric on the space $\text{Imm}([0, 2\pi], \mathbb{R}^2)/\text{trans}$ of plane curves modulo translations.

The metric $G^{a,b}$ is invariant with respect to the reparametrization group $\text{Diff}^+([0, 2\pi])$.

Proof: The pullback metric is defined via

$$G_c^{a,b}(h, h) = \langle T_c R(h), T_c R(h) \rangle_{L^2},$$

and hence we need to compute the derivative of the R -transform. First we compute the derivatives of the functions $c \mapsto |c'|^{1/2}$ and $c \mapsto v$, which are

$$T_c \left(|c'|^{1/2} \right) (h) = \frac{1}{2} \frac{\langle h', c' \rangle}{|c'|^{3/2}} = \frac{1}{2} |c'|^{1/2} \langle D_s h, v \rangle$$

$$T_c v(h) = \frac{1}{|c'|} \langle D_s h, v \rangle c' + \frac{1}{|c'|} h'$$

$$= D_s h - \langle D_s h, v \rangle v = \langle D_s h, n \rangle n.$$

Now computing the derivative of the R -transform is simple.

$$\begin{aligned} T_c R(h) &= \frac{1}{2} |c'|^{1/2} \langle D_s h, v \rangle \left(a \begin{pmatrix} v \\ 0 \end{pmatrix} + \sqrt{4b^2 - a^2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \\ &\quad + |c'|^{1/2} a \begin{pmatrix} \langle D_s h, n \rangle n \\ 0 \end{pmatrix} \\ &= |c'|^{1/2} \left(a \langle D_s h, n \rangle n + \frac{a}{2} \langle D_s h, v \rangle v \right) \\ &\quad + \frac{1}{2} \sqrt{4b^2 - a^2} \langle D_s h, v \rangle \end{aligned}$$

Therefore the pullback metric is

$$\begin{aligned} G_c^{a,b}(h, h) &= \int_0^{2\pi} \left(\left| a \langle D_s h, n \rangle n + \frac{a}{2} \langle D_s h, v \rangle v \right|^2 + \left(b^2 - \frac{a^2}{4} \right) \langle D_s h, v \rangle^2 \right) |c'| d\theta \\ &= \int_0^{2\pi} a^2 \langle D_s h, n \rangle^2 + b^2 \langle D_s h, v \rangle^2 ds. \end{aligned}$$

To obtain the formula for the operator $P_c^{a,b}$ we apply integration by parts to the above formula:

$$\begin{aligned} &\int_0^{2\pi} a^2 \langle D_s h, \langle D_s h, n \rangle n \rangle + b^2 \langle D_s h, \langle D_s h, v \rangle v \rangle ds \\ &= \left(a^2 \langle h, \langle D_s h, n \rangle n \rangle + b^2 \langle h, D_s \langle D_s h, v \rangle v \rangle \right) \Big|_0^{2\pi} \\ &\quad - \int_0^{2\pi} a^2 \langle h, D_s (\langle D_s h, n \rangle n) \rangle + b^2 \langle h, D_s (\langle D_s h, v \rangle v) \rangle ds \\ &= \left(a^2 \langle h, \langle D_s h, n \rangle n \rangle + b^2 \langle h, D_s \langle D_s h, v \rangle v \rangle \right) \Big|_0^{2\pi} \\ &\quad - \int_0^{2\pi} a^2 \langle h, \langle D_s^2 h, n \rangle n - \langle D_s h, \kappa v \rangle n - \langle D_s h, n \rangle \kappa v \rangle ds \\ &\quad - \int_0^{2\pi} b^2 \langle h, \langle D_s^2 h, v \rangle v + \langle D_s h, \kappa n \rangle v + \langle D_s h, v \rangle \kappa n \rangle ds. \end{aligned}$$

Here we used that $\kappa = \langle D_s v, n \rangle = -\langle D_s n, v \rangle$.

The metric $G^{a,b}$ is invariant under reparametrizations,

$$G_{c \circ \varphi}^{a,b}(h \circ \varphi, h \circ \varphi) = G_c^{a,b}(h, h),$$

since it is written in terms of operations D_s, v, n which are equivariant with respect to reparametrizations. \square

2.3 Image of the R -transform

To characterize the image of the R -transform we note that we can reconstruct c' from the first two components of $R(c)$ via

$$c' = \frac{1}{a^2} \sqrt{R_1^2(c) + R_2^2(c)} \begin{pmatrix} R_1(c) \\ R_2(c) \end{pmatrix}$$

and hence $a|c'|^{1/2} = \sqrt{R_1^2(c) + R_2^2(c)}$. This implies the relation

$$R_3(c) = \frac{\sqrt{4b^2 - a^2}}{a} \sqrt{R_1^2(c) + R_2^2(c)},$$

which can be written in the form

$$(4b^2 - a^2) (R_1(c)^2 + R_2(c)^2) = a^2 R_3(c)^2. \quad (1)$$

Let us define the following cone

$$C^{a,b} = \{q \in \mathbb{R}^3 : (4b^2 - a^2)(q_1^2 + q_2^2) = a^2 q_3^2, q_3 > 0\}$$

in \mathbb{R}^3 . Then the image of the R -transform consists of curves, which lie in $C^{a,b}$. That is for a curve $q \in C^\infty([0, 2\pi], \mathbb{R}^3)$ we have

$$\begin{aligned} q \in \text{im } R &\iff q(\theta) \in C^{a,b}, \forall \theta \in [0, 2\pi], \\ \text{im } R &= C^\infty([0, 2\pi], C^{a,b}). \end{aligned}$$

In the special case $4b^2 = a^2$ the $R^{2a,a}$ -transform has no third component and the image of the $R^{2a,a}$ -transform is the open set consisting of all curves, which avoid $(0, 0) \in \mathbb{R}^2$. The latter condition arises from the requirement that the curves be non-degenerate.

The inverse of the R -transform can be computed using the identity

$$c' = \frac{1}{2ab} |R| \begin{pmatrix} R_1(c) \\ R_2(c) \end{pmatrix}.$$

Therefore

$$\begin{aligned} R_1 : \text{im } R &\rightarrow \text{Imm}([0, 2\pi], \mathbb{R}^2) / \text{trans} \\ R_1(q)(\theta) &= p_0 + \frac{1}{2ab} \int_0^\theta |q(\theta)| \begin{pmatrix} q_1(\theta) \\ q_2(\theta) \end{pmatrix} d\theta. \end{aligned}$$

Of course the inverse of the R -transform is defined only up to translation, which manifests itself as the freedom to choose the starting point $p_0 \in \mathbb{R}^2$ of the integration.

The cone $C^{a,b}$ is a flat hypersurface in \mathbb{R}^3 , since there is a isometric covering map from the polar coordinate domain $\{(r, \varphi) : r > 0\}$ to $C^{a,b}$ given by

$$q(r, \varphi) = \begin{pmatrix} \frac{r}{m} \cos(m\varphi) \\ \frac{r}{m} \sin(m\varphi) \\ \frac{\sqrt{4b^2 - a^2}}{2b} r \end{pmatrix},$$

where we introduced the constant $m = \frac{2b}{a}$. A straightforward computation

$$(4b^2 - a^2)(q_1^2 + q_2^2) = \frac{4b^2 - a^2}{m^2} r^2 = a^2 q_3^2$$

checks that the image of this map is the cone $C^{a,b}$. From

$$\begin{aligned} dq_1^2 + dq_2^2 + dq_3^2 &= \frac{1}{m^2} dr^2 + r^2 d\varphi^2 + \frac{4b^2 - a^2}{4b^2} dr^2 \\ &= dr^2 + r^2 d\varphi^2 \end{aligned}$$

we see that the map $q(r, \varphi)$ is an isometry from the Euclidean metric in \mathbb{R}^2 , which has the expression $dr^2 + r^2 d\varphi^2$ in polar coordinates, to the natural metric on the cone $C^{a,b}$.

The inverse map is determined only up to a multiple of 2π and is given by

$$\Phi(q) = \begin{pmatrix} r(q) = \frac{2b}{\sqrt{4b^2 - a^2}} q_3 \\ \varphi(q) = \frac{a}{2b} \left(\arctan\left(\frac{q_2}{q_1}\right) + 2k\pi \right) \end{pmatrix}$$

with $k \in \mathbb{Z}$. Using the inverse map we can write the distance function on the cone

$$\begin{aligned} \text{dist}^2(q, \bar{q}) &= (x_1(q) - x_1(\bar{q}))^2 + (x_2(q) - x_2(\bar{q}))^2 \\ &= r(q)^2 + r(\bar{q})^2 - 2r(q)r(\bar{q})\cos(\varphi(q) - \varphi(\bar{q})) \\ &= \min_{k \in \mathbb{Z}} \frac{4b^2}{4b^2 - a^2} \left(q_3^2 + \bar{q}_3^2 - \right. \\ &\quad \left. - 2q_3\bar{q}_3 \cos \left(\frac{a}{2b} \left(\arctan \frac{q_2}{q_1} - \arctan \frac{\bar{q}_2}{\bar{q}_1} \right) + \frac{a}{b}k\pi \right) \right). \end{aligned} \quad (2)$$

The minimum appears, because the angle is only determined up to a multiple of 2π .

Theorem 2.4. *The metric $G^{a,b}$ on open curves is flat. Geodesics are the preimages under the R -transform of geodesics on the flat space $\text{im } R$.*

A path of curves $q : \mathbb{R} \times [0, 2\pi] \rightarrow C^{a,b}$ in $\text{im } R$ is a geodesic, if for each $\theta \in [0, 2\pi]$ the curve $t \mapsto q(t, \theta)$ is a geodesic in $C^{a,b}$.

The geodesic distance between $c, \bar{c} \in \text{Imm}([0, 2\pi], \mathbb{R}^2)$ is given by the integral over the pointwise distance,

$$\int_0^{2\pi} \text{dist}(R(c)(\theta), R(\bar{c})(\theta)) d\theta.$$

However the minimum over $k \in \mathbb{Z}$ is not to be taken pointwise, but only once for all values of θ . This corresponds to choosing a continuous lift of the curve $R(c)$ via Φ .

Proof: Since the cone $C^{a,b}$ is flat in the sense of Riemannian geometry, so is the space $C^\infty([0, 2\pi], C^{a,b})$ of curves in $C^{a,b}$ with respect to the L^2 -metric given by

$$G_q(h, k) = \int_0^{2\pi} \langle h, k \rangle d\theta,$$

for $q \in C^\infty([0, 2\pi], C^{a,b})$ and h, k tangent vectors at q . Note that this metric does not depend on the basepoint q . The metric is the same at all points in the space. Note also that the image of the R -transform

$$\text{im } R = C^\infty([0, 2\pi], C^{a,b}),$$

equals the set of curves, which lie in the cone $C^{a,b}$. It is a property of the L^2 -metric that geodesics in $\text{im } R$ are given by paths of curves $q(t, \theta)$, such that for each fixed θ , the curve $q(\cdot, \theta)$ is a geodesic in the cone $C^{a,b}$. The length of this geodesic will be given by an expression of the form (2) with some $k \in \mathbb{Z}$, not necessarily the smallest one. The length of the path $q(t, \theta)$ in $\text{im } R$ is given by the integral over the lengths of each point-wise path $q(\cdot, \theta)$. Since we have a continuous family of geodesics on $C^{a,b}$, the value for k will be the same for all $\theta \in [0, 2\pi]$. Hence the geodesic distance between two elements $q, \bar{q} \in \text{im } R$ is given by the integral

$$\int_0^{2\pi} \text{dist}(q(\theta), \bar{q}(\theta)) d\theta,$$

where the minimum over $k \in \mathbb{Z}$ is taken only once for all values of θ . \square

See [7, Theorem 9.1] for more details on the flat L^2 -metric.

3 THE R -TRANSFORM FOR CLOSED CURVES

In this section we want to consider the R -transform acting on closed curves. First note that Lemma 2.1 and Theorem 2.2 remain valid, if we replace open curves by closed ones.

Restricting our attention to closed curves $C^\infty(S^1, \mathbb{R}^2)$ means that we impose additional constraints on the image of the R -transform. From the inversion formula

$$R_1(q)(\theta) = \frac{1}{a^2} \int_0^\theta \sqrt{q_1(\theta)^2 + q_2(\theta)^2} \begin{pmatrix} q_1(\theta) \\ q_2(\theta) \end{pmatrix} d\theta,$$

we see that a curve q is the image of a closed curve only if the condition

$$F(q) = \int_0^\theta \sqrt{q_1(\theta)^2 + q_2(\theta)^2} \begin{pmatrix} q_1(\theta) \\ q_2(\theta) \end{pmatrix} d\theta = 0$$

is satisfied. Let us denote the image of the R -transform, restricted to closed curves, by

$$\mathcal{C}^{a,b} = \{q \in C^\infty(S^1, C^{a,b}) : F(q) = 0\}.$$

Note the difference between the cone $C^{a,b}$, which is a submanifold of \mathbb{R}^3 and the space $\mathcal{C}^{a,b}$, which is a submanifold in the space of curves.

Theorem 3.1. *The image $\mathcal{C}^{a,b}$ of the manifold of closed curves under the R -transform is a codimension 2 submanifold of the flat space $C^\infty(S^1, C^{a,b})$.*

A basis of the orthogonal complement $(T_q \mathcal{C}^{a,b})^\perp$ is given by the two vectors

$$\begin{aligned} U_1(q) &= \frac{1}{\sqrt{q_1^2 + q_2^2}} \begin{pmatrix} 2q_1^2 + q_2^2 \\ q_1 q_2 \\ 0 \end{pmatrix} + \frac{2}{a} \sqrt{4b^2 - a^2} \begin{pmatrix} 0 \\ 0 \\ q_1 \end{pmatrix}, \\ U_2(q) &= \frac{1}{\sqrt{q_1^2 + q_2^2}} \begin{pmatrix} q_1 q_2 \\ q_1^2 + 2q_2^2 \\ 0 \end{pmatrix} + \frac{2}{a} \sqrt{4b^2 - a^2} \begin{pmatrix} 0 \\ 0 \\ q_2 \end{pmatrix}. \end{aligned}$$

Proof: A basis of $(T_q \mathcal{C}^{a,b})^\perp$ can be computed by projecting the gradients of two components of the function $F = (F_1, F_2)$ to the tangent space of $C^\infty(S^1, C^{a,b})$. Let $q \in \mathcal{C}^{a,b}$ be a curve and $h \in C^\infty(S^1, \mathbb{R}^3)$ a tangent vector. Then

$$\begin{aligned} T_q F_1(h) &= \int_0^{2\pi} \frac{q_1 h_1 + q_2 h_2}{\sqrt{q_1^2 + q_2^2}} q_1 + \sqrt{q_1^2 + q_2^2} h_1 d\theta \\ T_q F_2(h) &= \int_0^{2\pi} \frac{q_1 h_1 + q_2 h_2}{\sqrt{q_1^2 + q_2^2}} q_2 + \sqrt{q_1^2 + q_2^2} h_2 d\theta. \end{aligned}$$

Thus the two gradients are

$$\begin{aligned} \text{grad}^{L^2} F_1(q) &= \frac{q_1}{\sqrt{q_1^2 + q_2^2}} \begin{pmatrix} q_1 \\ q_2 \\ 0 \end{pmatrix} + \sqrt{q_1^2 + q_2^2} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \\ \text{grad}^{L^2} F_2(q) &= \frac{q_2}{\sqrt{q_1^2 + q_2^2}} \begin{pmatrix} q_1 \\ q_2 \\ 0 \end{pmatrix} + \sqrt{q_1^2 + q_2^2} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}. \end{aligned}$$

Differentiating the governing equation for the cone (1) we obtain that the tangent space $T_q C^\infty(S^1, C^{a,b})$ is given by all curves h , which satisfy the pointwise condition

$$(4b^2 - a^2)(q_1 h_1 + q_2 h_2) = a^2 q_3 h_3. \quad (3)$$

A projection (not necessarily orthogonal) of the vectors $\text{grad}^{L^2} F_i(q)$ can be found by choosing λ_i such that the vector $\text{grad}^{L^2} F_i(q) + \lambda_i(0, 0, 1)^T$ satisfies (3). A simple computation shows that for the component F_i the right value is $\lambda_i = 2q_i \frac{\sqrt{4b^2 - a^2}}{a}$. This completes the proof. \square

We can now use the Gram-Schmidt procedure to compute an orthonormal base of $(T_q \mathcal{C}^{a,b})^\perp$, which is better suited for computational purposes. The formulas however do not reveal more structure.

3.2 The curvature of the space $\mathcal{C}^{2a,a}$

As mentioned previously the R -transform has no third component in the case $b = 2a$ and the cone $\mathcal{C}^{2a,a}$ reduces to the space $\mathbb{R}^2 \setminus 0$. Let $\widetilde{U}_1(q), \widetilde{U}_2(q) \in C^\infty(S^1, \mathbb{R}^2)$ be the orthonormal basis of $(T_q \mathcal{C}^{2a,a})^\perp$ derived via the Gram-Schmidt procedure from the basis $U_1(q), U_2(q)$. Using this ONB we can express the curvature of the cone $\mathcal{C}^{2a,a}$.

To do so we take a constant vector field $q \mapsto (q, h)$ on $C^\infty(S^1, \mathbb{R}^2)$ and its orthonormal projection

$$X_h(q) = h - \langle \widetilde{U}_1(q), h \rangle \widetilde{U}_1(q) - \langle \widetilde{U}_2(q), h \rangle \widetilde{U}_2(q) \in T_q \mathcal{C}^{2a,a}.$$

Then we take the flat covariant derivative in $\text{Imm}(S^1, \mathbb{R}^2)$

$$\bar{\nabla}_{X_h(q)} X_k(q) = D_{q, X_h(q)} X_k(q) = d(X_k(q))(X_h(q)).$$

The orthonormal projection of this vector field onto $(T_q \mathcal{C}^{2a,a})^\perp$ is then equal to the value of the second fundamental form $S \in (T_q \mathcal{C}^{2a,a})^\perp$, i.e.:

$$S(X_h(q), X_k(q)) = \langle d(X_k(q))(X_h(q)), \widetilde{U}_1(q) \rangle_{L^2} \widetilde{U}_1(q) + \langle d(X_k(q))(X_h(q)), \widetilde{U}_2(q) \rangle_{L^2} \widetilde{U}_2(q).$$

The curvature of $\mathcal{C}^{2a,a}$ at q is then given by the Gauß-equation [18, 26.4]:

$$\begin{aligned} \langle R(X_h(q), X_k(q))X_k(q), X_h(q) \rangle_{L^2} &= \\ &= -\|S(X_h(q), X_k(q))\|_{L^2}^2 \\ &+ \langle S(X_k(q), X_k(q)), S(X_h(q), X_h(q)) \rangle_{L^2} \\ &= -\langle d(X_h(q))(X_k(q)), \widetilde{U}_1(q) \rangle_{L^2}^2 \\ &- \langle d(X_k(q))(X_h(q)), \widetilde{U}_2(q) \rangle_{L^2}^2 \\ &+ \langle d(X_h(q))(X_h(q)), \widetilde{U}_1(q) \rangle_{L^2} \langle d(X_k(q))(X_k(q)), \widetilde{U}_1(q) \rangle_{L^2} \\ &+ \langle d(X_h(q))(X_h(q)), \widetilde{U}_2(q) \rangle_{L^2} \langle d(X_k(q))(X_k(q)), \widetilde{U}_2(q) \rangle_{L^2}. \end{aligned}$$

4 THE INDUCED METRIC ON SHAPE SPACE \mathcal{S}

In the rest of the paper we will mainly consider closed curves. However the results can be easily reformulated for the case of open curves.

The shape space \mathcal{S} denotes the space of unparametrized plane curves, which can be represented as the quotient $\mathcal{S} := \text{Imm}(S^1, \mathbb{R}^2)/\text{Diff}(S^1)$ of parametrized

curves modulo parametrizations. Associated to this quotient is the natural projection

$$\pi : \text{Imm}(S^1, \mathbb{R}^2) \rightarrow \text{Imm}(S^1, \mathbb{R}^2)/\text{Diff}(S^1),$$

which maps a curve c to its image $C = \pi(c)$.

By Theorem 2.2 the metric $G^{a,b}$ on $\text{Imm}(S^1, \mathbb{R}^2)$ is invariant with respect of the reparametrization group $\text{Diff}(S^1)$. Therefore there exists a unique Riemannian metric \bar{G} on shape space, such that the projection π is a Riemannian submersion. Associated to the projection π the decomposition of the tangent bundle $T\text{Imm}(S^1, \mathbb{R}^2)$ into horizontal and vertical parts. The vertical bundle Ver is the kernel of the projection π , i.e. $\text{Ver} = \ker T\pi$, and the horizontal bundle

$$\text{Hor}(c) = \text{Ver}(c)^\perp \subset T_c \text{Imm}(S^1, \mathbb{R}^2)$$

is defined as the orthogonal complement of Ver with respect to the Riemannian metric $G^{a,b}$. The action of $\text{Diff}(S^1)$ on $\text{Imm}(S^1, \mathbb{R}^2)$ induces an infinitesimal action of its Lie algebra $\mathfrak{X}(S^1)$, given by

$$\zeta_\mu(c) = c' \mu \in T_c \text{Imm}(S^1, \mathbb{R}^2)$$

with $\mu \in \mathfrak{X}(S^1)$. This defines a vector field ζ_μ on the space $\text{Imm}(S^1, \mathbb{R}^2)$. The vertical bundle consists of the image of all infinitesimal vector fields

$$\text{Ver}(c) = \{\zeta_\mu(c) : \mu \in \mathfrak{X}(S^1)\}.$$

When we fix a curve c , there is a one-to-one correspondence between the vector fields on the circle $\mathfrak{X}(S^1)$ and the space $\text{Ver}(c)$ given by the map $\mu \mapsto \zeta_\mu(c)$.

From the theory of Riemannian submersions [18] it follows that

- Geodesics on shape space with respect to \bar{G} correspond to horizontal geodesics on the manifold $\text{Imm}(S^1, \mathbb{R}^2)$ of parametrized curves with respect to $G^{a,b}$. Horizontal geodesics on $\text{Imm}(S^1, \mathbb{R}^2)$ are those, whose tangent vector lies in the horizontal bundle, i.e. $\partial_t c(t) \in \text{Hor}(c(t))$.
- The geodesic distance on shape space can be computed using the formula

$$\text{dist}(C_0, C_1) = \inf_{\varphi \in \text{Diff}(S^1)} \text{dist}(c_0, c_1 \circ \varphi),$$

where dist on the right hand side denotes the geodesic distance on $\text{Imm}(S^1, \mathbb{R}^2)$ with the $G^{a,b}$ -metric.

- The curvature of the shape space can be calculated using O'Neil's curvature formula, see for example [18]. Given two orthonormal vector fields X, Y on the space \mathcal{S} of unparametrized curves, the sectional curvature K is given by

$$K_{\mathcal{S}}(X, Y) = K_{\text{Imm}}(\tilde{X}, \tilde{Y}) + \frac{3}{4} |[\tilde{X}, \tilde{Y}]^{\text{vert}}|^2.$$

Here \tilde{X}, \tilde{Y} are horizontal lifts of the vector fields X, Y to $\text{Imm}(S^1, \mathbb{R}^2)$ and $[\tilde{X}, \tilde{Y}]^{\text{vert}}$ denotes the vertical projection of the vector field $[\tilde{X}, \tilde{Y}]$.

Theorem 4.1. *Consider the shape space of open curves $\mathcal{S} = \text{Imm}([0, 2\pi], \mathbb{R}^2)/\text{Diff}([0, 2\pi])$ with the metric that induces from the $G^{a,b}$ -metric on parametrized curves. The curvature of this space is non-negative.*

Proof: Theorem 2.4 shows that the space $\text{Imm}([0, 2\pi], \mathbb{R}^2)$ of parametrized curves is flat, which implies $K_{\text{Imm}}(\tilde{X}, \tilde{Y}) = 0$ in O’Neil’s curvature formula. The only remaining term

$$K_S(X, Y) = \frac{3}{4} |[\tilde{X}, \tilde{Y}]^{\text{vert}}|^2.$$

is clearly non-negative. \square

5 RELATING SHAPE TRANSFORMATIONS

In this section we describe a general method of constructing reparametrisation invariant metrics on the space of plane curves. This method will include the metrics studied by Srivastava et al. [23], Mio et al. [20], the analogue for curves of the surface metric from Kurtek et al. [10, 11] and the method of Younes et al. [27] as special cases.

Let us consider a general transform $F : \text{Imm}(S^1, \mathbb{R}^2) \rightarrow C^\infty(S^1, \mathbb{R}^n)$ mapping plane curves to curves in \mathbb{R}^n for some $n \in \mathbb{N}$. The transform F induces a reparametrisation invariant metric, if it satisfies an equivariance property.

Lemma 5.1. *If $F : \text{Imm}(S^1, \mathbb{R}^2) \rightarrow C^\infty(S^1, \mathbb{R}^n)$ satisfies the equivariance property*

$$F(c \circ \varphi) = \varphi'^{\frac{1}{2}} F(c) \circ \varphi, \quad (4)$$

with $c \in \text{Imm}(S^1, \mathbb{R}^2)$ and $\varphi \in \text{Diff}(S^1)$, then the pullback by F of the flat L^2 -metric on $C^\infty(S^1, \mathbb{R}^n)$ is invariant under the reparametrization group $\text{Diff}(S^1)$.

Proof: Let us denote the pullback metric by G^F . It is given via the formula

$$G_c^F(h, h) = \int_{S^1} |T_c F(h)|^2 d\theta.$$

The infinitesimal version of the equivariance property is

$$T_{c \circ \varphi} F(h \circ \varphi) = \varphi'^{\frac{1}{2}} T_c F(h) \circ \varphi$$

and therefore we see that

$$\begin{aligned} G_{c \circ \varphi}^F(h \circ \varphi, h \circ \varphi) &= \int_{S^1} |T_{c \circ \varphi} F(h \circ \varphi)|^2 d\theta \\ &= \int_{S^1} |T_c F(h) \circ \varphi|^2 |\varphi'| d\theta = G_c^F(h, h), \end{aligned}$$

i.e. the pullback is invariant under $\text{Diff}(S^1)$. \square

If additionally the function F is infinitesimally injective, i.e. $T_c F$ is injective for all $c \in \text{Imm}(S^1, \mathbb{R}^2)$, then the two-form G^F is positive definite and F thus induces a reparametrisation invariant Riemannian metric on the manifold of immersed curves.

5.2 The Q -transform

Another member in this family of shape transformations is the Q -transform, which was introduced in Mani et al. [14] for curves and in Kurtek et al. [10, 11] for surfaces. It has properties similar to the R -transform, and yet is much more difficult to study.

Following [14] let us define for a curve $c \in \text{Imm}(S^1, \mathbb{R}^2)$ the Q -transform by

$$\begin{aligned} Q : \text{Imm}(S^1, \mathbb{R}^2) &\rightarrow C^\infty(S^1, \mathbb{R}^2) \\ Q(c) &= |c'|^{\frac{1}{2}} c. \end{aligned}$$

First let us note the following properties of the Q -transform.

Lemma 5.3. *For a curve $c \in \text{Imm}(S^1, \mathbb{R}^2)$ and a diffeomorphism $\varphi \in \text{Diff}(S^1)$ we have:*

- *Equivariance under reparametrizations*

$$Q(c \circ \varphi) = |\varphi'|^{1/2} \cdot (Q(c) \circ \varphi)$$

- *Behavior under translations*

$$Q(c + p) = Q(c) + |c'|^{1/2} p \text{ for } p \in \mathbb{R}^2$$

- *Scaling property*

$$Q(\rho \cdot c) = \rho^{3/2} \cdot Q(c) \text{ for } \rho \in \mathbb{R}_{>0}$$

- *Equivariance under rotations*

$$Q(e^{i\alpha} c) = e^{i\alpha} Q(c) \text{ for } \alpha \in S^1.$$

Proof: This lemma can be proven by direct calculations similar to Lemma 2.1. \square

Note that the behavior of the Q -transform under translation of the curve is rather unintuitive. The curve is being translated by a vector, which depends on the local length of the curve.

Theorem 5.4. *The Q -transform induces a reparametrization invariant metric on $\text{Imm}(S^1, \mathbb{R}^2)$, given by*

$$G_c(h, k) = \int_{S^1} \langle h + \frac{1}{2} \langle D_s h, v \rangle c, k + \frac{1}{2} \langle D_s k, v \rangle c \rangle ds. \quad (5)$$

Proof: We need to show that the metric is non-degenerate, i.e. that $G_c(h, h) = 0$ only when $h = 0$. This condition is equivalent to the transform Q being infinitesimally injective, i.e. $T_c Q \cdot h = 0$ only when $h = 0$. In the proof of Theorem 2.2 we computed

$$T_c \left(|c'|^{1/2} \right) (h) = \frac{1}{2} |c'|^{1/2} \langle D_s h, v \rangle$$

and therefore we have

$$T_c Q \cdot h = |c'|^{1/2} \left(h + \frac{1}{2} \langle D_s h, v \rangle c \right). \quad (6)$$

Since we are only considering immersions for which $|c'| \neq 0$, the condition $T_c Q \cdot h = 0$ is equivalent to

$$\langle h', c' \rangle c + 2|c'|^2 h = 0. \quad (7)$$

For all θ , where $c(\theta) = 0$ this already implies $h(\theta) = 0$. On the open set $\{\theta : c(\theta) \neq 0\}$ we expand the tangent vector h into a part that points along c and a part orthogonal to

it, $h = h_1 c + h_2 c^\perp$ with $c \perp c^\perp$. Then (7) can be rewritten as

$$\langle h', c' \rangle c + 2|c'|^2 h_1 c + 2|c'|^2 h_2 c^\perp = 0.$$

This implies that $h_2 = 0$ everywhere and we are left with the equation

$$\langle c, c' \rangle h'_1 + 3|c'|^2 h_1 = 0. \quad (8)$$

As before for θ where $\langle c(\theta), c'(\theta) \rangle = 0$, we infer that $h_1(\theta) = 0$. It remains to study on the open set, where neither $c(\theta)$ nor $\langle c(\theta), c'(\theta) \rangle$ vanish. This open set is the union of disjoint open intervals. Let us denote one such interval by (θ_0, θ_1) . On this interval every solution of (8) is given by

$$h_1(\theta) = C e^{\int_{\tilde{\theta}}^{\theta} \frac{-3|c'|^2}{\langle c, c' \rangle} d\theta},$$

with $\tilde{\theta} \in (\theta_0, \theta_1)$. Since we are looking for one smooth solution h_1 on S^1 and we already know that for θ_0 and θ_1 the solution has to satisfy $h_1(\theta_0) = h_1(\theta_1) = 0$, because these points lie outside the open set, we see that only the solution with $C = 0$ can satisfy this. Therefore $h_1 \equiv 0$ and with it $h \equiv 0$ on all of S^1 .

The expression (5) for the metric $G_c(h, k)$ is an immediate consequence of (6). The reparametrization invariance of the metric is either a consequence of Lemma 5.1 or can be seen directly from (5). \square

The injectivity of the map $T_c Q$ is essential in order to use the metric for shape comparisons. However we could not find a proof for this anywhere in the literature.

Questions that were comparably easy to answer for the R -transform are much more difficult for the Q -transform. To our knowledge it is not known how image of the Q -transform on either open or closed curves looks like; whether it is open or a smooth submanifold of the space $C^\infty(S^1, \mathbb{R}^2)$. Even finding a numerically efficient way to invert the Q -transform presents difficulties.

The main reason for these difficulties is that the Q -transform scales the curve c with the object $|c'|^{1/2}$, which geometrically is the square-root of the volume-element on the curve and should multiply tangent vectors along c , not the curve itself.

5.5 The transform of Younes et al. [27]

In this section we will show that the method applied in Younes et al. [27] to study a Sobolev-type metric on curves also fits into the setup described in this paper. The basic mapping considered in [27] is given by

$$\begin{aligned} \Phi : C^\infty([0, 2\pi], \mathbb{R}^2 \setminus 0) &\rightarrow \text{Imm}([0, 2\pi], \mathbb{R}^2) \\ \Phi(q)(\theta) &= \frac{1}{2} \int_0^\theta \begin{pmatrix} q_1^2(u) - q_2^2(u) \\ 2q_1(u)q_2(u) \end{pmatrix} du. \end{aligned}$$

The basic mapping has the property that it pulls back the Sobolev metric of order one

$$G_c(h, h) := \int_0^{2\pi} \langle D_s h, D_s h \rangle ds$$

to the flat L^2 -metric on the space $C^\infty([0, 2\pi], \mathbb{R}^2 \setminus 0)$. To fit this mapping into our framework we define the R -transform as the inverse of Φ ,

$$\begin{aligned} R : \text{Imm}([0, 2\pi], \mathbb{R}^2) &\rightarrow C^\infty([0, 2\pi], \mathbb{R}^2 \setminus 0) \\ R(c) &= \sqrt{|c'|} \begin{pmatrix} \cos \frac{\alpha}{2} \\ \sin \frac{\alpha}{2} \end{pmatrix}, \end{aligned}$$

where $\alpha(\theta)$ is the turning angle of c defined via

$$c'(\theta) = |c'(\theta)| (\cos \alpha(\theta), \sin \alpha(\theta)).$$

Theorem 5.6 (From [27]). *The R -transform induces the Sobolev metric of order one*

$$G_c(h, h) := \int_0^{2\pi} \langle D_s h, D_s h \rangle ds$$

on the space of parametrized curves.

6 NUMERICAL COMPUTATION OF GEODESICS

In this section we will describe a way to numerically compute the shortest path in shape space of closed curves between two shapes C_0 and C_1 . We will do so for the special case $4b^2 = a^2 = 1$. In this case the R -transform maps plane curves into plane curves,

$$R : \text{Imm}([0, 2\pi], \mathbb{R}^2) \rightarrow C^\infty([0, 2\pi], \mathbb{R}^2), \quad R(c) = |c'|^{1/2} v.$$

The cone $C^{a,a/2}$ regarded as a subset of \mathbb{R}^2 simplifies to $C^{a,a/2} = \mathbb{R}^2 \setminus 0$. Therefore the image of the R -transform of open curves is the set

$$\text{im } R = C^\infty([0, 2\pi], \mathbb{R}^2 \setminus 0)$$

of all curves, which avoid the origin, which is an open subset of the space of all curves. Restricting ourselves to the closed curves, we obtain from Theorem 3.1 that the image \mathcal{C} of the R -transform is a codimension 2 submanifold of $C^\infty(S^1, \mathbb{R}^2)$.

6.1 Geodesics between parametrized curves

Geodesics in the preshape space $\text{Imm}(S^1, \mathbb{R}^2)$ of parametrized curves correspond under the R -transform to geodesics on \mathcal{C} with the Riemannian metric induced from the flat L^2 -metric on $C^\infty(S^1, \mathbb{R}^2)$.

Using Theorem 3.1 we can implement a projection operator which, given a curve $q \in \mathcal{C}$ and a tangent vector p to q , computes its orthogonal projection onto $T_q \mathcal{C}$.

```
function PROJ(q, p)
  U1, U2 ← ONB of (Tqℳ)⊥
  return p - ⟨p, U1⟩ U1 - ⟨p, U2⟩ U2
end function
```

The forward computation of the geodesic from a curve $q \in \mathcal{C}$ with initial velocity $p \in T_q \mathcal{C}$ can be seen as a constrained optimisation problem. A geodesic on \mathcal{C} is a minimum of

$$\int_0^1 \frac{1}{2} |u(t)|^2 + \langle p(t), \dot{q}(t) - u(t) \rangle + \lambda(t) \cdot F(q(t)) dt.$$

Since we use the Euclidean metric on the ambient space, we will not distinguish between velocities $u(t)$ and momenta $p(t)$. The variable $\lambda(t) \in \mathbb{R}^2$ is a Lagrange multiplier, which enforces the constraint

$$F(q) = \int_{S^1} |q(\theta)| q(\theta) d\theta,$$

defining the cone $\mathcal{C} = F_1(0)$. These are holonomic constraints and there are already several methods, that can solve this problem; see e.g. [12] for an overview. We have chosen to use the RATTLE algorithm [1], which is a symplectic integrator that preserves the constraints through an explicit projection of the velocity vector, so that it lies on the constraint surface. The variational equations in continuous time have the form

$$\begin{aligned} \dot{q}(t) &= p(t) \\ \dot{p}(t) &= \lambda(t) \cdot \nabla F(q(t)). \end{aligned}$$

We denote by N the number of time-steps used by the integrator, so $\Delta t = \frac{1}{N}$.

function EXP(q, p, N)

```

 $q_0, p_0 \leftarrow q, p$ 
for  $i \leftarrow 0, N-1$  do
   $\bar{p} \leftarrow p_i + \frac{\Delta t}{2} \lambda \cdot \nabla F(q_i)$ 
   $q_{i+1} \leftarrow q_i + \Delta t \bar{p}$ 
  estimate initial value for  $\lambda$  to enforce constraint
  iteratively adapt  $\lambda$  so that  $q_{i+1} \in \mathcal{C}$ 
   $p_{i+1} \leftarrow \bar{p} + \frac{\Delta t}{2} \mu \cdot \nabla F(q_{i+1})$ 
   $\mu$  chosen such that  $p_{i+1} \in T_{q_{i+1}} \mathcal{C}$ 
end for
return  $q, p$ 
end function

```

The next step is to solve the boundary value problem for parametrized shapes. Given two curves $q_0, q_1 \in \mathcal{C}$ we need to find the initial velocity p , such that the endpoint $\text{Exp}_{q_0} p$ is close to q_1 . We do so via a fixed-point iteration, using the fact, that our space \mathcal{C} is a submanifold of a flat space.

function LOG(q_0, q_1, N)

```

 $p \leftarrow \Delta t \text{PROJ}(q_0, q_1 - q_0)$ 
 $\tilde{q} \leftarrow \text{EXP}(q_0, p, 1)$ 
 $p \leftarrow \text{PROJ}(q_0, \tilde{q} - q_0)$ 
while  $|\text{EXP}(q_0, p, N) - q_1| > \varepsilon$  do
   $\tilde{q} \leftarrow \text{EXP}(q_0, p, N)$ 
   $p \leftarrow p + \alpha \text{PROJ}(q_0, q_1 - \tilde{q})$ 
end while
return  $p$ 
end function

```

To construct the initial guess for p we project the straight line between q_0 and q_1 to $T_{q_0} \mathcal{C}$. This would be a valid initial guess. Based on experiments, however we found that computing one step of the forward-shooting algorithm to obtain \tilde{q} and projecting the straight line $\tilde{q} - q_0$ back to $T_{q_0} \mathcal{C}$ leads to a better initial guess. The iteration consists of computing the endpoint $\tilde{q} = \text{Exp}_{q_0} p$ of the geodesic with initial velocity p and updating p in the direction $q_1 - \tilde{q}$.

The parameter α controls the step-size of the iteration. It can be either fixed for the whole minimization or chosen adaptively. In our experiments we chose α as large as possible, while still ensuring that the distance decreased, compared to the last iteration.

It is also possible to view this iteration as an approximation to a gradient descent algorithm on the geodesic distance

$$E(p) = \frac{1}{2} \text{dist}(\text{Exp}_{q_0} p, q_1)^2.$$

The derivative of this function is

$$T_p E(\delta p) = G_{\text{Exp}_{q_0} p} \left(-\text{Log}_{\text{Exp}_{q_0} p} q_1, T_p \text{Exp}_{q_0}(\delta p) \right).$$

If we approximate the logarithm by the straight line, $\text{Log}_{\text{Exp}_{q_0} p} q_1 \approx q_1 - \text{Exp}_{q_0} p$, the differential of the exponential map by the identity, $T_p \text{Exp}_{q_0}(\delta p) \approx \delta p$ and the parallel transport along the geodesic $t \mapsto \text{Exp}_{q_0}(tp)$ from $\text{Exp}_{q_0} p$ to q_0 by the projection to $T_{q_0} \mathcal{C}$, then we obtain

$$T_p E(\delta p) \approx G_{q_0} (-\text{Proj}_{q_0} (q_1 - \text{Exp}_{q_0} p), \delta p)$$

and hence the approximation of the gradient is given by

$$\nabla_p E \approx -\text{Proj}_{q_0} (q_1 - \text{Exp}_{q_0} p).$$

6.2 Finding the optimal parametrization

As explained in Section 4, finding geodesics between unparametrized shapes corresponds to finding horizontal geodesics or equivalently finding the minimum of

$$\text{dist}(C, D) = \inf_{\psi \in \text{Diff}(S^1)} \text{dist}(c, d \circ \psi),$$

with $C = \pi(c)$, $D = \pi(d)$ and where dist on the right denotes the geodesic distance on parametrized curves, computed as described in Section 6.1. We will compute this minimum using a gradient descent algorithm on the function

$$E(\varphi) = \frac{1}{2} \text{dist}(c, d \circ \varphi)^2. \quad (9)$$

We use $d \circ \varphi_1$ instead of $d \circ \varphi$, in order to have a left action of $\text{Diff}(S^1)$ on the space of curves. We want to compute the right-trivialized gradient $\nabla_\varphi E$ of E , defined as

$$\langle \nabla_\varphi E, \mu \rangle_{\mathfrak{X}(S^1)} = T_\varphi E(\mu \circ \varphi)$$

for $\mu \in \mathfrak{X}(S^1)$ and for a choice of an inner product $\langle \cdot, \cdot \rangle_{\mathfrak{X}(S^1)}$ on $\mathfrak{X}(S^1)$.

From Riemannian geometry we know that the gradient of

$$G(p) = \frac{1}{2} \text{dist}(p, q)^2$$

is given by

$$\nabla_p G = -\text{Log}_p q.$$

Hence

$$T_\varphi E(\delta \varphi) = G_{d \circ \varphi_1} (-\text{Log}_{d \circ \varphi_1} c, -(d \circ \varphi_1)' \delta \varphi \circ \varphi_1)$$

and by writing $\delta \varphi = \mu \circ \varphi$ for some $\mu \in \mathfrak{X}(S^1)$ we get

$$T_\varphi E(\mu \circ \varphi) = G_{d \circ \varphi_1} (\text{Log}_{d \circ \varphi_1} c, (d \circ \varphi_1)' u).$$

After introducing an inner product on $\mathfrak{X}(S^1)$, we could compute the gradient of $E(\varphi)$ by solving

$$\langle \nabla_\varphi E, u \rangle_{\mathfrak{X}(S^1)} = G_{d \circ \varphi_1}(\text{Log}_{d \circ \varphi_1} c, (d \circ \varphi_1)' u).$$

There is however a better expression for the gradient, which fits better with the action of $\text{Diff}(S^1)$ on $\text{Imm}(S^1, \mathbb{R}^2)$.

Consider the related function

$$\bar{E}(\varphi) = \frac{1}{2} \text{dist}(c \circ \varphi_1, d)^2.$$

Its gradient can be computed using the invariance of the metric under φ , i.e. $\bar{E}(\varphi) = E(\varphi_1)$, which implies

$$\begin{aligned} T_\varphi \bar{E}(\delta\varphi) &= T_{\varphi_1} E(-(\varphi_1)' \delta\varphi \circ \varphi_1) \\ &= -T_{\varphi_1} E\left(\left(\frac{1}{\varphi_1'} \delta\varphi\right) \circ \varphi_1\right) \\ &= G_{d \circ \varphi} \left(-\text{Log}_{d \circ \varphi} c, (d \circ \varphi)' \frac{1}{\varphi_1'} \delta\varphi\right) \\ &= G_{d \circ \varphi}(-\text{Log}_{d \circ \varphi} c, (d' \circ \varphi) \delta\varphi). \end{aligned}$$

Using the invariance of Log and the metric G_c under reparametrizations,

$$\begin{aligned} \text{Log}_{d \circ \varphi} c \circ \varphi &= (\text{Log}_d c) \circ \varphi \\ G_{d \circ \varphi}(h \circ \varphi, k \circ \varphi) &= G_d(h, k), \end{aligned}$$

we get

$$\begin{aligned} T_\varphi \bar{E}(\delta\varphi) &= G_{d \circ \varphi}(-(\text{Log}_d(c \circ \varphi_1)) \circ \varphi, (d' \circ \varphi) \delta\varphi) \\ &= G_d(-\text{Log}_d(c \circ \varphi_1), d'(\delta\varphi \circ \varphi_1)), \end{aligned}$$

and hence the gradient can be obtained by solving

$$\langle \nabla_\varphi \bar{E}, \mu \rangle_{\mathfrak{X}(S^1)} = -G_d(\text{Log}_d(c \circ \varphi_1), d' \mu).$$

Since the functions E and \bar{E} differ only by exchanging c and d , we have another way to express the gradient of E ,

$$\langle \nabla_\varphi E, \mu \rangle_{\mathfrak{X}(S^1)} = -G_c(\text{Log}_c(d \circ \varphi_1), c' \mu). \quad (10)$$

As the inner product on the space of vector fields we can use the one induced by the identification of $\mathfrak{X}(S^1)$ with the vertical space at the curve c using the infinitesimal action $\zeta_\mu(c) = c' \mu$. The inner product is given by

$$\langle \mu, \nu \rangle_{\mathfrak{X}(S^1)} = G_c(c' \mu, c' \nu). \quad (11)$$

Theorem 6.3. *The (right-trivialized) gradient of the energy*

$$E(\varphi) = \frac{1}{2} \text{dist}(c, d \circ \varphi_1)^2$$

with respect to the inner product (11) is given by the vector field on S^1 corresponding to the vertical projection of $-\text{Log}_c(d \circ \varphi_1)$, i.e.

$$c' \nabla_\varphi E = -\text{Ver}_c(\text{Log}_c(d \circ \varphi_1)).$$

It can be computed by solving the equation

$$G_c(c' \nabla_\varphi E, c' \mu) = -G_c(\text{Log}_c(d \circ \varphi_1), c' \mu)$$

for $\nabla_\varphi E \in \mathfrak{X}(S^1)$ with $\mu \in \mathfrak{X}(S^1)$ being an arbitrary vector field.

Proof: Note that the equation

$$G_c(h, c' \mu) = -G_c(\text{Log}_c(d \circ \varphi_1), c' \mu)$$

is satisfied for all $\mu \in \mathfrak{X}(S^1)$ if and only if

$$h = -\text{Ver}_c(\text{Log}_c(d \circ \varphi_1)).$$

Each element in the vertical space corresponds via the infinitesimal action to one element of $\mathfrak{X}(S^1)$. The theorem now follows from combining equations (10) and (11). \square

The next algorithm computes the element $\mu \in \mathfrak{X}(S^1)$ corresponding to the orthogonal projection of a vector $h \in T_c \text{Imm}(S^1, \mathbb{R}^2)$ to the vertical subspace $\text{Ver}(c)$.

```
function VER( $c, h$ )
   $\nu \leftarrow$  test function on  $\mathfrak{X}(S^1)$ 
   $\mu \leftarrow$  solution of  $G_c(c' \mu, c' \nu) = G_c(h, c' \nu)$ 
  return  $\mu$ 
end function
```

We use a finite element method with Lagrange elements of first order to numerically compute the vertical projection. A more explicit formula for the inner product $G_c(h, k)$ is given by

$$G_c(h, k) = \int_{S^1} \frac{\langle h', k' \rangle}{|c'|} - \frac{3}{4} \frac{\langle h', c' \rangle \langle k', c' \rangle}{|c'|^3} d\theta.$$

The algorithm to find geodesics between unparametrized shapes C, D takes as input two parametrizations c, d of these shapes such that $\pi(c) = C$ and $\pi(d) = D$ and finds the diffeomorphism $\psi \in \text{Diff}(S^1)$, such that the geodesic distance $\text{dist}(c, d \circ \psi)$ is minimal.

```
function SOLVEBVP( $c, d, N$ )
   $\psi \leftarrow \text{Id}_{S^1}$   $\triangleright$  Notation  $\psi := \varphi_1$  in (9)
  while  $\text{dist}(c, d \circ \psi)$  is not minimal do
     $h \leftarrow \text{LOG}(c, d \circ \psi, N)$ 
     $\mu \leftarrow \text{VER}(c, h)$   $\triangleright$  Notation  $\mu := -\nabla_\varphi E$ 
     $\eta \leftarrow \text{FLOW}(-\mu, \text{Id}_{S^1}, \alpha)$ 
     $\psi \leftarrow \psi \circ \eta$ 
  end while
  return  $\psi$ 
end function
```

To understand the algorithm note that ψ corresponds to φ_1 in Theorem 6.3. There is no need to compute φ itself, as only φ_1 is necessary to compute the reparametrization of the curve d . In each iteration of the algorithm we first compute the gradient $\mu = -\nabla_\varphi E$ with the help of Theorem 6.3. A continuous gradient descent would take the form

$$\partial_t \varphi = -\nabla_\varphi E \circ \varphi.$$

A first order time-discretization would be

$$\varphi_{i+1} = \text{Fl}^\mu(\alpha, \text{Id}_{S^1}) \circ \varphi_i$$

with $\text{Fl}^\mu(\alpha, \text{Id}_{S^1})$ denoting the flow of the vector field $\mu = -\nabla_\varphi E$ up to time α starting from Id_{S^1} at time 0.

Since we are only interested in $\psi = \varphi_1$, we can rewrite this as

$$\begin{aligned}\psi_{i+1} &= \varphi_{i+1} = \varphi_i \circ \text{Fl}^\mu(\alpha, \text{Id}_{S^1})_1 \\ &= \psi_i \circ \text{Fl}^{-\mu}(\alpha, \text{Id}_{S^1}).\end{aligned}$$

The gradient descent step is repeated until the relative decrease of $\text{dist}(c, d \circ \psi)$ falls below a prescribed threshold.

Similarly to the function $\text{Log}_{q_0}(q_1)$ in Section 6.1, the parameter α is the step-size for the gradient-descent and is chosen adaptively to ensure the distance decreasing in each step.

6.4 Adaptive grid refinement

The behavior of shortest paths between unparametrized curves can be understood as a combination of stretching and bending. Bending of a parametrized curve is numerically well behaved, since uniformly sampled curve will stay approximately uniformly sampled. Stretching however tends to expand a very short section of the curve into a much larger one and will lead to a curve that is under-sampled in the expanded area, unless the original parametrization was chosen to counter this effect. See Figure. 2 for an example of this.

In the gradient descent algorithm from Section 6.2 we start with a curve that is sampled uniformly at the points

$$x_0 = 0, x_1 = \frac{2\pi}{n}, \dots, x_{n-1} = \frac{n-1}{n}2\pi.$$

At each iteration, given a grid x_0, \dots, x_{n-1} for the curve c , the curve d would be sampled at the points $\psi(x_0), \dots, \psi(x_{n-1})$. Stretching of the curve c would correspond to a large derivative $\psi'(x)$ or a large distance $|\psi(x_{i+1}) - \psi(x_i)|$ between two consecutive points. We add points to the grid, whenever the distance exceeds that of a uniform grid, i.e.

$$|\psi(x_{i+1}) - \psi(x_i)| > \frac{2\pi}{n_0}.$$

Here n_0 is the size of the original grid. By this we ensure that the target curve $d \circ \psi$ will not be under-sampled.

Similarly we remove a point x_i , whenever the neighbouring points would be sufficient to provide enough resolution, i.e. the conditions

$$|x_{i+1} - x_{i-1}| < \frac{2\pi}{n_0} \text{ and } |\psi(x_{i+1}) - \psi(x_{i-1})| < \frac{2\pi}{n_0}$$

are satisfied.

7 RESULTS

In this section we present experiments demonstrating our numerical computations. The algorithms described in Section 6 were implemented in Python using the NumPy and SciPy libraries. To compute the vertical projection of a tangent vector as described in Section 6.2 we used the finite element library FEniCS [13]. The shapes used for the experiments come from the database

of closed binary shapes collected by the LEMS Vision Group at Brown university (<http://www.lems.brown.edu/~dmc>). Each curve was initially parametrized by a set of 300 points positioned at equal arc length along the curve. Each geodesic between curves with fixed parametrizations was computed using 25 time steps. To find the optimal parametrization the gradient descent algorithm from Section 6.2 with a maximum of 100 iterations was used.

In Table 1 we show the geodesic distance between several shapes, the number of iterations necessary to find the optimal reparametrization and the number of points on the discretized curve. Figure 1 shows examples of some of the geodesics. The geodesics are sampled at timesteps 0, 5, 10, 15, 20 and 25. The first and last images in each row thus show the template and target curves respectively. While the template curve is always parametrized proportional to arc length, the target curve will in general have an irregular parametrization, while still representing the same shape.

From a mathematical point of view we would expect the distance between two shapes to be symmetric, i.e. $d(C_0, C_1) = d(C_1, C_0)$. However, if we look at the table, we see that the distance from the cat to the dog is about 20% smaller than the distance from the dog to the cat. The difference is even more striking in the distances between the cow and the dog. The reason for this difference can be found by looking at the minimizing geodesics. The optimal geodesic from the dog to the cow is shrinking a leg of the dog and at the same time growing another leg of the cow from the body. As we can see in Figure 3 such growth tends to originate from a single point. Thus one point of the dog will be expanding to create the whole leg of the cow, while the whole leg of the dog will collapse to one point on the cow. This is why it is necessary to dynamically add points to the discretization of the template and target curves as described in Section 6.4.

In Figure 2 we show a more pronounced example of the pathological behaviour of the metric. We compute the geodesic between the two shapes in the first row of Figure 2. On the left hand side we can see the minimal geodesic computed using the grid refinement method described in Section 6.4. The lines connecting the two shapes show the paths of the marked points along the geodesic. We see that the whole fold of the target shape is being created out of a single point of the template shape. For comparison we also tried to solve the shape matching problem without grid refinement. The result can be seen in the right image of the bottom row. The template shape is lacking the resolution to resolve the fold and therefore the matching is rather unsatisfactory.

8 CONCLUSIONS

Riemannian metrics on shape space of plane curves are of great interest in a wide variety of applications in image analysis and computational anatomy. Due to

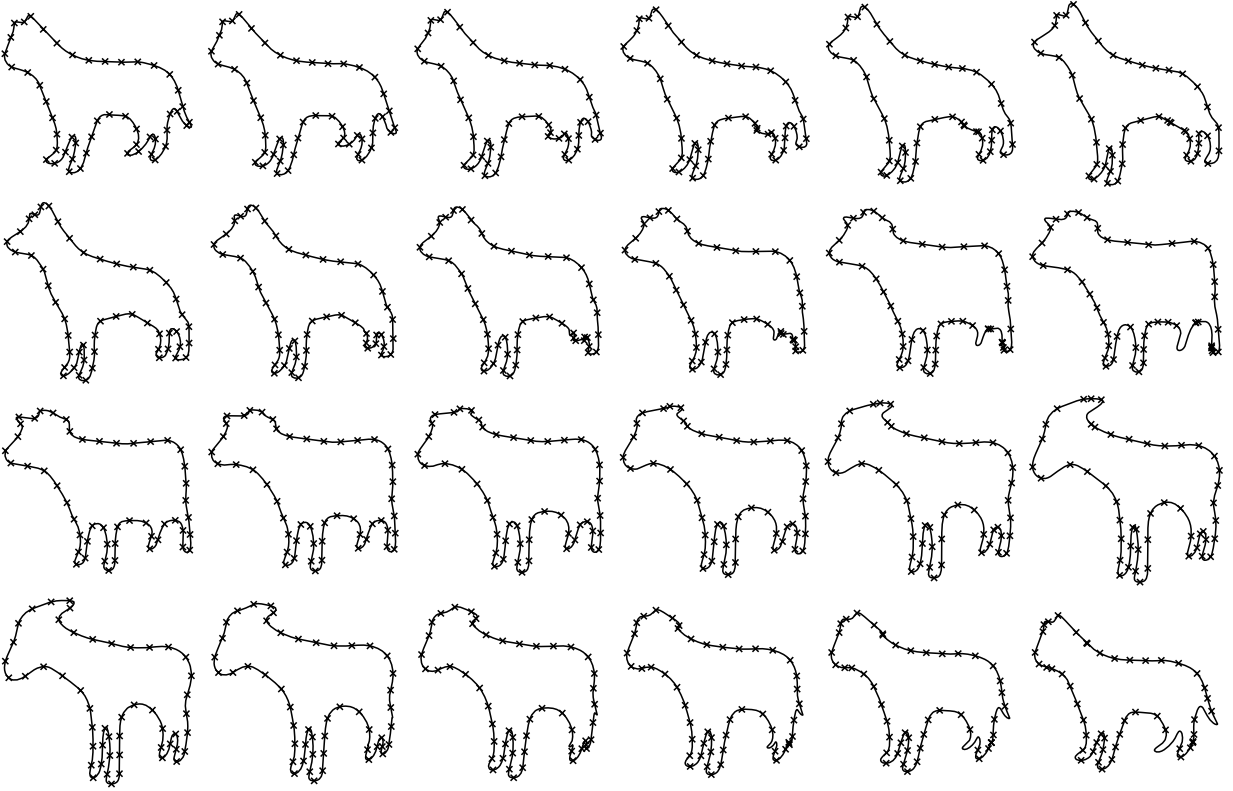


Fig. 1. Examples of geodesics between several shapes. The geodesics shown are from the cat to the dog in the first row, from the dog to the cow in the second, from the cow to the donkey in the third and from the donkey back to the cat in the last row thus forming a geodesic quadrangle. The marked points on each geodesic are the paths of initially equidistantly spaced points of the template curve showing the stretching and compression along minimal geodesics.

TABLE 1

Geodesic distances between several shapes as computed by the minimization algorithm. The number of iterations needed to achieve the minimum and the number of points on the refined grid are also shown. From a numerical point of view the distinction between template and target is significant. On average the distances differ by about 14%.

I_1	I_2	$I_1 \rightarrow I_2$			$I_2 \rightarrow I_1$			%diff
		# iterations	# points	distance	# iterations	# points	distance	
cat	cow	28	456	7.339	33	462	8.729	15.9
cat	dog	36	475	8.027	102	455	10.060	20.2
cat	donkey	73	476	12.620	102	482	12.010	4.8
cow	dog	45	460	6.225	97	466	9.705	35.8
cow	donkey	32	452	7.959	26	511	7.915	0.6
dog	donkey	15	457	8.299	10	476	8.901	6.8
shark	airplane	63	491	13.741	40	487	13.453	2.1

the infinite dimensional nature of shape space, metrics that allow for efficient computations of geodesics and distance between shapes are particularly useful.

In this paper we generalize the R -transform, first introduced in the work of [23], to the class of all elastic metrics, whose coefficients satisfy $4b^2 \geq a^2$, where a and b are parameters controlling the degree of bending and stretching of the curve respectively (see Section 1.1). This transformation allows us to obtain efficient algorithms for computing the geodesic distance between shapes as well as to gain a better understanding of the geometry of the space. For the case of open curves we obtain explicit formulas for geodesics and we can show that the space

of parametrized open curves is a flat space in the sense of Riemannian geometry. As a consequence, it follows that the shape space of unparameterized open curves has positive sectional curvature. For closed curves the situation is more difficult, since the space of parametrized curves is not flat anymore, but the representation is still very useful from a numerical point of view.

We have presented experiments showing the minimal geodesics between closed, unparametrized curves and computed the geodesic distances between them. For certain pairs of shapes the R -transform allows for very efficient computations, while for other pairs we see that the tendency of the metric to compress and stretch points

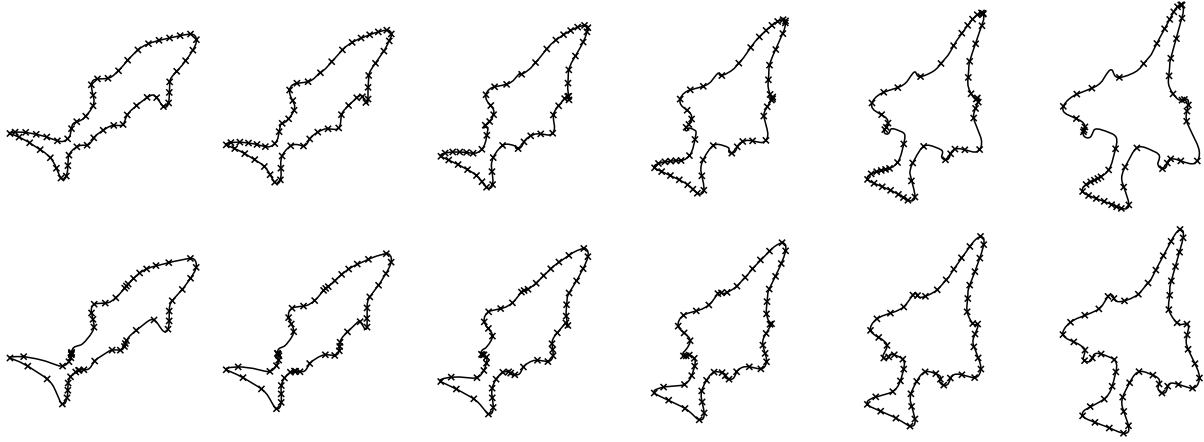


Fig. 3. The top rows shows the minimal geodesic from the shark and the plane. The bottom row shows the minimal geodesic computed with template and target curves reversed. We see that the intermediate shapes almost coincide.

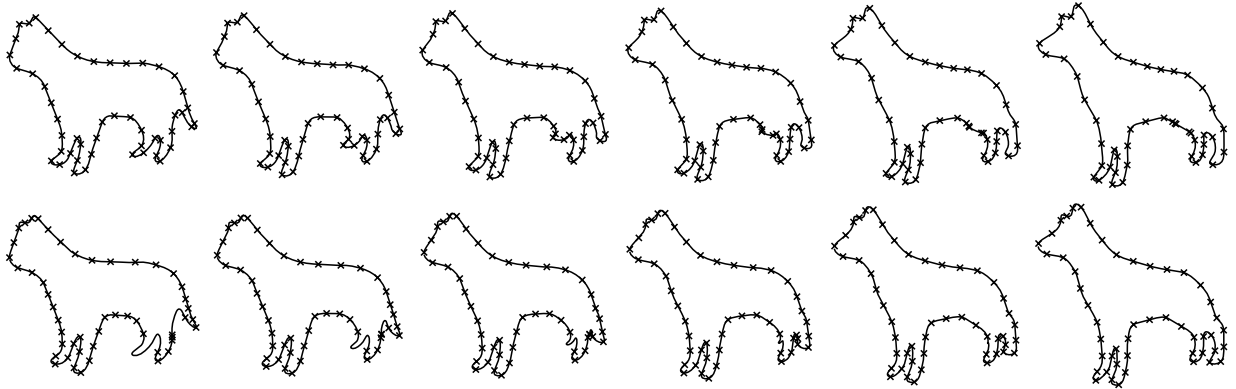


Fig. 4. Similarly as in Figure 3 the top row shows the minimal geodesic from the cat to the dog. The bottom row shows the minimal geodesic computed with template and target curves reversed. Since in this case there is stretching and compression happening along the geodesics the two paths are not symmetric any more. This also corresponds to the difference in the geodesic distances as can be seen in Table 1

along the curve leads to numerical difficulties. In particular the difference in the geodesic distance, depending on which curve was chosen as template and which as target shows that more work is required to understand this behaviour and to develop numerical methods capable of dealing with this situation. We plan to address these questions in future work.

Another way to address this problem would be to use metrics of higher order, i.e. those that depend on higher derivatives of either the curve c or the variation h . Finding efficient representations of such metrics similar to the R -transform used for metrics of order one remains an open question.

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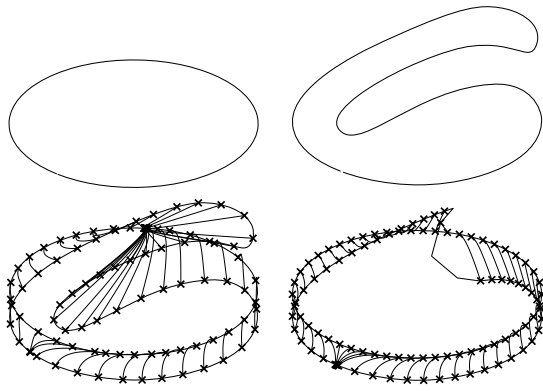


Fig. 2. This image shows the minimal geodesic between the two shapes in the first row. The template shape is an ellipse while the target shape is an ellipse with a large fold. The bottom left image shows the minimal geodesic computed with the grid adaptively refined as in Section 6.4. In the bottom right image the geodesic is computed without grid refinement, which leads to a loss of resolution along the fold. Since the fold is growing out of a point it is necessary to refine the grid in the neighborhood of this point to accurately capture the fold.

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